Stationary and dynamic critical behavior of the contact process on the Sierpinski carpet

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I. INTRODUCTION

Nonequilibrium phase transitions often show universal scaling phenomena [1–3]. Nonequilibrium systems may be considered as natural dynamical extensions of static universality classes established in equilibrium, although Markovian models, such as reaction-diffusion models, present second-order phase transitions that are not related to equilibrium models. We are particularly interested in systems that present a phase transition between an absorbing state and an active statistically stationary state. A mean-field modeling is able to capture the critical exponents in high dimensions. However, density fluctuations in low-dimensional systems invalidate this approach, and stochastic models have to be explicitly employed [4]. For most dynamic transitions with a single absorbing state, the directed percolation (DP) universality class is the appropriate one. If the preferred direction is time, the system presents a phase transition even in one spatial dimension. If time is not the preferred direction, one must consider at least two-dimensional models.

The critical behavior of the contact process, which involves self-replication and spontaneous annihilation, is related to the DP universality class. One example is a Markov chain in which each site of a lattice has two states A and B. A turns into B according to a given rate depending on the state of its neighbors. One of the states, say, A, does not appear spontaneously. Therefore the B (vacuum state) is an absorbing state, and it is reached if the lifetime of A is small enough. A statistically stationary state with a finite population of the A state is reached above a critical lifetime [5].

In the present contribution, we study a model of interacting populations whose critical behavior places the model in the DP universality class. In particular we investigate the threshold of coexistence of the A and B population in the Sierpinski carpet fractal. It is well known that the critical behavior of equilibrium systems in fractal lattices may depend not only on the effective dimensionality but also on several intrinsic topological properties [6]. In nonequilibrium phase transitions, the coupling between the dynamic evolution and the underlying scale-invariant topology may also lead to the emergence of an unconventional critical behavior. Our approach will be the one based on stochastic spatially structured models. In the last few years, a great number of works have shown the relevance of this kind of approach to describe biological population problems [7–15]. We focus on the stochastic lattice model for a contact process system introduced by Harris [5]. This model exhibits a phase diagram with active states where both species coexist and an absorbing state where one of the species has been extinct.

Some of the dynamic critical exponents associated to the asynchronous version of this model have been obtained in Ref. [16]. It was found that one of the exponents deviates significantly from the interpolation lines of the regular-lattice results for the case of the Sierpinski fractal. For $d = \log 8 / \log 3 \approx 1.893$ the scaling exponent $\beta = 0.706$ was obtained, larger than $\beta = 0.583$ which holds for regular two-dimensional lattices. On the other hand, for the checkerboard fractal, the reported results fully agreed with the interpolation lines. As pointed out in Ref. [17] from a qualitative point of view, when decreasing the space dimension, the critical exponents associated with the singularities of the mean cluster size and the correlation length increase while the order parameter exponent $\beta$ decreases.

In Ref. [18] the Sierpinski fractal was studied, and the dynamical critical exponents $\delta$ and $\eta$, related respectively to the survival probability and order parameter critical relaxation, also followed the interpolation lines, while the value of the mean-square spreading distance exponent $Z$ does not follow the interpolation lines of the regular lattices.
author attributed the difference to the use of an Euclidean measure since two points on a fractal which are close in Euclidean space could be very far in the path connecting these points. Nevertheless, the results are in agreement with the scaling relation \( dZ = 4\delta + 2\eta \), \( d \) being the fractal dimension.

Discrete scale invariance (DSI) was found in the same fractal media [19]. The authors showed that the physical observables became coupled to the topology and lead to a logarithm-oscillatory modulation to the power laws. It was conjectured that, for the exponents \( \eta \) and \( \delta \), the exact relationship \( \delta + \eta = (d + 2)/6 \) could hold for integer dimensions \( 2 \leq d \leq 4 \). The logarithmic-periodic modulations in the physical observables appears only when the epidemics is initialized at the same single site. When averaging results starting from randomly selected sites, the oscillations became severely damped and could be confused with statistical noise. These oscillations were not observed in Ref. [18] because, in order to reduce this effect, the author made an average over different initial positions of the seed particle. The obtained \( Z \) does not follow the interpolation value, but he obtained the expected \( \theta \) and \( \delta \) exponents. In Ref. [20], for the Sierpinski carpet in the Ising model, an systematic decrease was observed in the dynamic exponent \( Z \) when the segmentation step was increased.

Here we will advance on the characterization of the contact process absorbing state phase transition on the Sierpinski fractal by providing a high-precision numerical study, together with finite-size and short-time dynamics scaling analysis, to determine both stationary and dynamical critical behavior. In particular, we will provide a highly accurate estimate of the threshold between the active and absorbing state by exploring the scale invariance of the relative order parameter fluctuations at the critical point. We will also give reliable estimates of the stationary critical exponents related to correlation length, the order parameter, and its fluctuations which interpolate between the \( d = 1 \) and \( d = 2 \) values, contrasting with the behavior found for the dynamic critical exponent \( Z \).

II. THE MODEL

We denote by \( X \) and \( Y \), respectively, the healthy and sick individuals. The habitat where the individuals interact and proliferate is represented here by a fractal lattice known as the deterministic Sierpinski carpet. This fractal lattice is constructed iteratively with the central site of each \( 3 \times 3 \) elementary block removed at each iteration step. A disordered version of the Sierpinski carpet would consider the removal of one site chosen at random from each elementary cell. For equilibrium phase transitions, there are numerical evidences of distinct universality classes on deterministic and random Sierpinski carpets [21]. Disorder may also strongly influence the critical behavior of nonequilibrium phase transitions [22]. However, it has been recently reported that a random distribution of local connectivities, as present on the Voronoi triangulation, is not able to change the directed percolation critical behavior in two dimensions [23]. If lattice disorder is a relevant perturbation for the direct percolation, critical behavior in fractal lattices is still an open issue. In the following, we will consider only the case of the deterministic Sierpinski carpet in order to explore the possible influence of the lattice scale invariance on the static and dynamic critical behavior.

FIG. 1. (Color online) Configuration at \( t = 560 \) Monte Carlo steps for the contact-process model on a Sierpinski fractal lattice with \( L = 3^4 \) sites and at \( p = 0.7 \). Infected sites are in red (dark gray), noninfected are in white. Black spaces stand for the forbidden sites of the Sierpinski fractal.

In the contact process, each site of the lattice (except for those which are the holes of the carpet) can be empty or occupied by just one individual of each species. Therefore, a site in the lattice can be in one of two states: occupied by a sick individual or occupied by a healthy individual. The model includes the following set of reactions:

\[
X + Y \xrightarrow{p} 2Y, \quad Y \xrightarrow{1-p} X, \quad (1)
\]

which describe the cyclic process \( X \rightarrow Y \rightarrow X \). These are the basic and relevant reactions that characterize a simple healthy-infected system. We take them into account here by considering the stochastic fractal lattice model defined by an asynchronous global dynamics composed of a set of local Markovian rules:

FIG. 2. (Color online) Density of infected individuals \( \psi \) versus \( p \) for distinct \( K \) generations in the vicinity of the absorbing state transition.
FIG. 3. (Color online) The moment ratio $m_{L}(p)$ as a function of the infection probability $p$ for distinct $K$ generations. The scale invariance at the critical point allowed us to precisely estimate the critical infection probability $p_c = 0.682875(5)$.

(a) Infection: In a site having a healthy individual, a sick one can be born with a probability $p/4$ times the number of sites occupied by sick individuals in the closest neighborhood. $p$ is a parameter that represents the infection probability of a site with four infected neighbors.

(b) Cure: In a site occupied by an infected individual, a healthy individual can be born with probability $q = 1 - p$, with $0 < p < 1$.

The critical behavior will be characterized by measuring a set of relevant static and dynamic critical exponents obtained by the use of a finite-size scaling analysis of the critical order parameter and its relative fluctuations. In what follows, we show results from simulations on finite lattices with $N = 8^K$ sites ($K$ is the generation number of the fractal). Each lattice sweep is considered as the time unit or one Monte Carlo step (MCS). The process is updated sequentially. We start from an initial condition with all sites in the fractal lattice covered by infected individuals avoiding the holes of the fractal. Once the system is placed in the initial condition, we apply the local rules (a) and (b). An example of a configuration obtained by simulations is shown in Fig. 1. The system evolves in time and eventually reaches a statistically stationary state. To avoid the system to become trapped in the absorbing state, we activate an infected individual in a randomly chosen site, avoiding the prohibited sites of the fractal, whenever the system falls in the absorbing state (reflective boundary condition) [24].

We measure the order parameter, the density of infected individuals $\psi(p,L) = \langle N_{y}(p,L) \rangle / N$, where $\langle N_{y}(p,L) \rangle$ is the average number of infected individuals in a lattice with linear size $L$, in the stationary regime as a function of $p$.

III. RESULTS

In the following results concerning the statistically stationary state, we have made all calculations after discarding

FIG. 4. Log-log plot of the order parameter versus the linear size $L$ at the critical point. From the best fit to a power law, we estimate the critical exponent ratio $\beta/\nu_{\perp} = 0.726(8)$.

FIG. 5. The logarithmic derivative of the order parameter versus $L$ at the critical point. From the best fit to a power law, we estimate the critical exponent $\nu_{\perp} = 0.78(1)$ in the Sierpinski carpet.

FIG. 6. (Color online) Order-parameter fluctuations $\Delta\psi(p,K)$ versus $p$ for distinct $K$ generations. The peak in the vicinity of the critical density signalizes the enhancement of the order-parameter fluctuations near the transition.
$L^2$ runs for relaxation. Typical averages ranged from $n = 6 \times 10^7$ configurations for the smallest lattices and $10^7$ for the largest ones. Such sampling of the statistically stationary state was large enough to obtain accurate measures of the order parameter, as well as of its moment and fluctuation, even close to the critical point where finite-size scaling will be employed. In Fig. 2, we show the density of infected individuals $\psi$, as a function of the infection probability $p$ obtained from simulations on lattices of distinct $K$ generation and linear size $L = 3^K$. As $L \to \infty$ a transition from a state with nonzero density of infected individuals to the absorbing state takes place by decreasing the values of $p$. The values used in our simulations were $0.6826 < p < 0.685$ with step $0.00001$ near the critical point. To precisely locate the critical infection probability $p_c$, we measured the relative fluctuation, that is, the ratio between the second moment and squared first moment of the number of infected individuals, defined as

$$m_L(p) = \frac{\langle N^2 \rangle}{\langle N \rangle^2},$$

which is roughly independent of the system size at the critical point. In Fig. 3, we plot $m_L(p)$ obtained from simulations performed in distinct lattice sizes (different $K$ generations), which allows us to estimate the critical probability as $p_c = 0.682875(5)$ from the crossing point of the two largest sizes used. An estimate based on the extrapolation of the slight size dependence of the different crossing points falls within the error bar. Once having located the critical point, finite-size scaling relations were used to compute the critical exponents characterizing such nonequilibrium phase transition. In particular, the order parameter obeys the power law $\psi(p_L, L) \propto L^{\beta/\nu_perp}$ and its logarithmic derivative $d \log \psi(p_L, L)/dp \propto L^{1/\nu_perp}$. These scaling laws are depicted in Figs. 4 and 5 from which we estimate $\beta/\nu_perp = 0.73(1)$ and $\nu_perp = 0.79(1)$. In Fig. 6, we report our data for the order parameter fluctuations,

$$\Delta \psi(p, L) = \left[\langle N^2 \rangle - \langle N \rangle^2 \right]/N,$$

for the Sierpinski fractal lattice versus $p$ for several lattices (different $K$ generation), which, in the vicinity of the critical point, scales as $\Delta \psi(p, L) \propto (p - p_c)^{-\gamma'}$. The increasing peaks signal the diverging fluctuations at the critical point.

### Table I

<table>
<thead>
<tr>
<th>Lattice</th>
<th>$\beta/\nu_perp$</th>
<th>$\nu_perp$</th>
<th>$\gamma'/\nu_perp$</th>
<th>$\beta$</th>
<th>$\gamma'$</th>
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<td>1.096</td>
<td>0.495</td>
<td>0.276</td>
<td>0.543</td>
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<td>Sierpinski</td>
<td>0.726(8)</td>
<td>0.78(1)</td>
<td>0.42(2)</td>
<td>0.576(3)</td>
<td>0.31(1)</td>
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<tr>
<td>Carpet</td>
<td>0.796(9)</td>
<td>0.733(7)</td>
<td>0.409(1)</td>
<td>0.583</td>
<td>0.299</td>
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<td>CP 2D</td>
<td>1.385</td>
<td>0.581</td>
<td>0.327</td>
<td>0.805</td>
<td>0.19</td>
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FIG. 7. (Color online) Log-log plot of the order-parameter fluctuations $\Delta \psi$ versus $L$ at the critical point. The exponent ratio $\gamma'/\nu_perp = 0.42(2)$ is estimated from the slope of the fitted straight line.

FIG. 8. Log-log plot of the order parameter $\psi(K = 7)$ versus $p - p_c$ above the critical point. The critical exponent $\beta$ is estimated from the slope of the fitted straight line $\psi \approx (p - p_c)^\beta$ from which we obtained $\beta = 0.576(3)$.

FIG. 9. Log-log plot of the order-parameter fluctuations $\Delta \psi$ versus $p - p_c$. The critical exponent $\gamma'$ is estimated from the slope of the fitted straight line $\Delta \psi \approx (p - p_c)^{-\gamma'}$ from which we obtained $\gamma' = 0.31(1)$. 

TABLE I. Values of critical exponents $\beta/\nu_perp$, $\nu_perp$, $\gamma'/\nu_perp$, $\beta$, and $\gamma'$ for the square lattice and Sierpinski fractal. CP results for integer dimensions were taken from Ref. [25].
as the thermodynamic limit is approached. The data for the order parameter fluctuations at the critical point are used in Fig. 7 to obtain the critical exponents ratio $\gamma'/\nu_\perp$ since $\Delta\psi(p_c) \propto L^{1/\nu_\perp}$ at the critical point.

In Figs. 8 and 9 we present independent calculations for the order parameter and fluctuations of the order parameter exponents from $\psi_k(p_c) \approx (p - p_c)^\beta$ and $\Delta\psi_k(p_c) \approx (p - p_c)^{-\gamma}$ obtained from generation $K = 7$ in the vicinity of the critical point $p_c$. In Table I we present the values of $\beta/\nu_\perp$, $\nu_\perp$, and $\gamma'/\nu_\perp$ for the Sierpinksi carpet and compare them with results for regular lattices with $d = 1, 2,$ and 3 dimensions. The exponents for the Sierpinski fractal with $1 < d < 2$ interpolates between those of the corresponding integer dimensions. Our results for $\beta/\nu_\perp$ and $\gamma'/\nu_\perp$ give for the hyperscaling relation $2\beta/\nu_\perp + \gamma'/\nu_\perp = d \approx 1.87$, which is consistent with $d = \log 8 / \log 3$.

We also probed the critical relaxation dynamics towards the statistically stationary state. Starting from a configuration with all sites in the active state, the time evolution of the order parameter, its moment ratio and fluctuations can be used to estimate some dynamical critical exponents. In what follows we considered the relaxation process in lattices with $K = 8$ generations and averaged the quantities over $n = 100 (K = 8)$ distinct runs. Calculations were performed up to $t = 10^5$.

The order parameter density relaxes towards its stationary value following the dynamic scaling law $\psi(p_c,t) \propto t^{-\beta/Z\nu_\perp}$, where $Z = \nu_\perp / \nu_\parallel$. In Fig. 10 we show such time evolution. The total run time $t = 10^5$ used in the dynamical simulation is smaller than the typical time needed to reach the absorbing state $t = L^d$ close to the transition.

In Fig. 11 we present an independent calculation of the transition point by the standard deviation and correlation coefficient method. The results of this method lead to $p_c = 0.68285(1)$ at which the value of the standard deviation of the data from the best power-law fitting presents a minimum while the correlation coefficient presents a maximum. This result is consistent with the value calculated by the cumulant technique. The exponent ratio $\beta/Z\nu_\perp$ is reported in Table II.

The dynamic scaling hypothesis also predicts that the moment ratio $m_L(p_c,t) \propto t^{d/Z}$. In Fig. 12 we report the time evolution of the order parameter moment ratio. The estimated value of the dynamic exponent $Z$ is included in Table II (considering $d = \log 8 / \log 3$). We emphasize that the estimated value is larger than those holding in Euclidean $d = 1$

### Table II. Values of critical exponents $\beta/Z\nu_\perp$, $Z$, $\gamma'/Z\nu_\perp$, $\beta/\nu_\perp$, $\gamma'/\nu_\perp$, and $\nu_\perp$ as obtained from the short-time dynamics scaling. CP results for the integer dimensions were taken from Ref. [25].

<table>
<thead>
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<th>$\beta/Z\nu_\perp$</th>
<th>$Z$</th>
<th>$\gamma'/Z\nu_\perp$</th>
<th>$\beta/\nu_\perp$</th>
<th>$\gamma'/\nu_\perp$</th>
<th>$\nu_\perp$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CP 1D</td>
<td>0.1594</td>
<td>1.58</td>
<td>0.314</td>
<td>0.252</td>
<td>0.496</td>
</tr>
<tr>
<td>Sierpinski carpet</td>
<td>0.39(1)</td>
<td>1.88(5)</td>
<td>0.24(2)</td>
<td>0.73(5)</td>
<td>0.45(5)</td>
</tr>
<tr>
<td>CP 2D</td>
<td>0.4505</td>
<td>1.76</td>
<td>0.231</td>
<td>0.793</td>
<td>0.406</td>
</tr>
<tr>
<td>CP 3D</td>
<td>0.73</td>
<td>1.90</td>
<td>0.172</td>
<td>1.387</td>
<td>0.327</td>
</tr>
</tbody>
</table>

FIG. 10. (Color online) Time evolution of the order-parameter at the critical point for $K = 8$, averaged over 100 runs. The dynamic scaling law $\psi(p_c,t) \propto t^{-\beta/Z\nu_\perp}$ holds in the short-time regime. Our best estimate of this critical exponent ratio is reported in Table II.

FIG. 11. Independent determination of the critical point from the short-time dynamics for $K = 8$ by using the standard deviation (main frame) and correlation coefficient (inset) method. When plotted against $p$, these quantities reach extremal values at the transition.

FIG. 12. Time evolution of the moment ratio $m_L(p_c,t)$ at the critical point. Same conditions as in Fig. 10. The dynamic scaling law $m_L(p_c,t) \propto t^{d/Z}$ holds in the short-time regime. Our estimate of this critical exponent ratio is reported in Table II.
and $d = 2$ lattices. These two dynamical exponents were used to obtain an independent estimate of the stationary exponent ratio $\beta/\nu_\perp$, which agrees with the value reported in Table I within the error bars.

The time evolution of the order parameter fluctuations is shown in Fig. 13. Dynamical scaling predicts $\Delta \psi(p_c, t) \propto t^{\gamma/\nu_\perp}$. The estimated critical exponent $\gamma/\nu_\perp$ is reported in Table II. This exponent can also be used to obtain an independent estimate of the stationary exponent ratio $\gamma/\nu_\perp$, also agreeing with the value reported in Table I within the error bars. In Table II we also report the exponents $\beta/\nu_\perp$ and $\gamma/\nu_\perp$ as obtained from the short-time dynamics scaling.

In Fig. 14 we present the result of the exponent $\nu_\parallel = Z_{\nu_\perp}$ calculated from $\frac{d \log(\psi(p_c, t))}{dp} \propto t^{1/\nu_\parallel}$.

**IV. CONCLUDING REMARKS**

We have investigated the critical behavior of a stochastic spatially structured model in which infected and healthy individuals reside on the sites of a Sierpinski fractal lattice and are described by discrete stochastic variables. The model presents a transition from an active state to an absorbing one at a critical infection probability $p_c$ which belongs to the directed percolation universality class. From numerical simulations of this irreversible model defined in such a fractal lattice and using finite-size scaling analysis, we computed relevant critical exponents governing this nonequilibrium phase transition. The results for the static critical exponents related to the vanishing of the order parameter as the transition threshold is approached, as well as those associated with the divergence of the order parameter fluctuations and correlation length, were shown to interpolate between the known values in $d = 1$ and $d = 2$ Euclidean lattices. On the other hand, the dynamic critical exponent related to the slow increase of the correlation length when the system evolves towards the absorbing state at criticality was shown to be larger than in both $d = 1$ and $d = 2$ lattices. This result corroborates previous findings indicating that the presence of forbidden regions of many size scales slows the critical relaxation dynamics as compared to the dynamics in regular lattices. This finding indicates that other topological properties, rather than the fractal dimension of the lattice, influence the dynamical critical behavior of nonequilibrium phase transitions. Therefore, one can anticipate that scale-free and regular lattices having the same box counting dimensionality may be governed by distinct dynamical exponents, although sharing a set of static critical exponents. This hypothesis can be tested by performing studies in generalized scale-free lattices with tunable fractal dimension. A systematic study along this direction would contribute to elucidate the topological quantities that are relevant to establish the universality class of absorbing state transitions in complex lattices.

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