A note on the volume form in normal matrix space

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Abstract

We present a local parametrization of the Riemann volume form in normal matrix space in terms of the spectral variables. Our method is direct and algebraic taking advantage of the unitary invariance of the ensemble and the Lie algebra of the unitary group.

Consider the volume Riemann volume $d_v M$ in the space of all $n \times n$ normal matrix. We wish to write the Riemann volume $d_v M$ in terms of the spectral coordinates.

A $n \times n$ matrix M is said to be normal if and only if it is a unitary similar to a diagonal matrix of its eigenvalues [3]. This means that the set of the eigenvectors of M form a mutually orthonormal eigenbasis on $\mathbb{C}^n$. If U is the matrix of eigenvectors, then M has a spectral representation

$$M = U \Lambda U^*, \tag{1}$$

where $\Lambda = diag\left[z_1, \cdots, z_n\right]$ is a diagonal matrix of its eigenvalues. This decomposition is unique up to n phases because the normalization of the eigenvectors of M. Therefore, we may write $U \in \mathcal{U}(n) \backslash \mathbb{T}^n$, where $\mathcal{U}(n)$ is the unitary group and $\mathbb{T}^n = \{e^{i\theta_1}, \cdots, e^{i\theta_n}\}$ is the n-torus.

A $n \times n$ normal matrix has n^2 complex entries (and hence $2n^2$ real entries) and $2d = n^2 + n$ independent real elements and from now we consider the $2d$ real upper triangular elements of M. Furthermore, given a unitary matrix U, we may use the representation

$$U = e^W. \tag{2}$$

where W is a skew-hermitian matrix $W = -W^*$. For our case, W will be a skew-hermitian which zero diagonal. This is to equalize the degrees of freedom of U. Usually for normal random matrices the coordinate change from the entries to the eigenvalues is done by evoking results from differential geometry [1]. No explicit computation of the Jacobian of the transformation is realized.

In this note, we perform the coordinate change by a direct and algebraic approach. Taking advantage of the unitary invariance of the ensemble and the Lie algebra of the unitary group, we explicitly evaluate the Jacobian of the coordinate change. We shall prove the following

Teorema 1 *Let M be a normal matrix of order n. Consider the spectral decomposition $M = U\Lambda U^*$ as in Eq. (1), with $U = e^W$ and W being a skew-hermitian which zero diagonal elements. Consider the change of coordinates*

$$M \xrightarrow{\varphi} (\Lambda, W) \tag{3}$$

the spectral variables (Λ, W) give rise to a local coordinate system in a subset of full measure. Furthermore, we have locally the Jacobian of this coordinate changes reads

$$\det J(\varphi) = \prod_{1 \le i < j \le n} |z_i - z_j|^2, \tag{4}$$

The above Theorem implies that the volume form reads as

$$d_v M = \prod_{1 \le i < j \le n} |z_i - z_j|^2 d_v \Lambda d_v W. \tag{5}$$

This representation of the volume form plays a major role in the eigenvalue statistics in normal random matrices [1, 5, 6].

1 Complex Coordinate Changes

In this note we explicitly use the isomorphism $\mathbb{C}^d \simeq \mathbb{R}^{2d}$ Suppose that some region of $\mathbb{C}^d$ has defined on it two systems of complex coordinates $\{m_k, \overline{m}_k\}_{k=1}^d$ and $\{w_k, \overline{w}_k\}_{k=1}^d$

We can regard coordinate m_k ($\overline{m}_k$) as a function of $\{w_k, \overline{w}_k\}_{k=1}^d$:

$$m_k = m_k\left(w_1, w_2, \cdots, w_d, \overline{w}_1, \overline{w}_2, \cdots, \overline{w}_d\right)$$

$$\overline{m}_k = \overline{m}_k\left(w_1, w_2, \cdots, w_d, \overline{w}_1, \overline{w}_2, \cdots, \overline{w}_d\right).$$

Computing the differential of the function m_k ($1-$form) with respect to the variables $\{w_k, \overline{w}_k\}_{k=1}^d$:

$$dm_k = \frac{\partial m_k}{\partial w_k} dw_1 + \frac{\partial m_k}{\partial w_2} dw_2 + \cdots + \frac{\partial m_k}{\partial \overline{w}_{d-1}} d\overline{w}_{d-1} + \frac{\partial m_k}{\partial \overline{w}_d} d\overline{w}_d \tag{6}$$

$$d\overline{m}_k = \frac{\partial \overline{m_k}}{\partial w_k} dw_1 + \frac{\partial \overline{m_k}}{\partial w_2} dw_2 + \cdots + \frac{\partial \overline{m}_k}{\partial \overline{w}_{d-1}} d\overline{w}_{d-1} + \frac{\partial \overline{m}_k}{\partial \overline{w}_d} d\overline{w}_d, \tag{7}$$

the partial derivatives in Eqs. (6,7) are the coordinates of the differential of the function m_k ($\overline{m}_k$) in the basis $\{dw_k, d\overline{w}_k\}_{k=1}^d$ of the space of the differentials in $\mathbb{C}^d \simeq \mathbb{R}^{2d}$.

We wish to define the Jacobian of the complex coordinate change $\{m_k, \overline{m}_k\}_{k=1}^d \to \{w_k, \overline{w}_k\}_{k=1}^d$. The Jacobian of this transformation is the determinant of the matrix whose entries are the coordinates of the differential of the functions m_k ($\overline{m}_k$) on the columns:

$$J = \begin{pmatrix} \frac{\partial m_1}{\partial w_1} & \cdots & \frac{\partial m_d}{\partial w_1} & \frac{\partial \overline{m}_1}{\partial w_1} & \cdots & \frac{\partial \overline{m}_d}{\partial w_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial m_1}{\partial w_d} & \cdots & \frac{\partial m_d}{\partial w_d} & \frac{\partial \overline{m}_1}{\partial w_d} & \cdots & \frac{\partial \overline{m}_d}{\partial w_d} \\ \frac{\partial m_1}{\partial \overline{w}_1} & \cdots & \frac{\partial m_d}{\partial \overline{w}_1} & \frac{\partial \overline{m}_1}{\partial \overline{w}_1} & \cdots & \frac{\partial \overline{m}_d}{\partial \overline{w}_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial m_1}{\partial \overline{w}_d} & \cdots & \frac{\partial m_d}{\partial \overline{w}_d} & \frac{\partial \overline{m}_1}{\partial \overline{w}_d} & \cdots & \frac{\partial \overline{m}_d}{\partial \overline{w}_d} \end{pmatrix}. \tag{8}$$

2 Random Normal Matrices *Ensembles*

Let $GL(n)$ denote the general linear algebra of all $n \times n$ complex matrices. We endow $GL(n)$ with the Euclidian structure of Frobenius

$$\langle M, N \rangle = Tr(MN^*),\ M, N \in GL(n).$$

Our goal is to compute the infinitesimal volume measure d_vM on the space of normal matrices. First, we need the two side unitary invariance of the ensemble.

Lemma 2 *Consider the space of the normal matrices of order n with inner product*

$$\langle A, B \rangle = \mathbf{Tr}\left(A^*B\right),$$

and define the linear maps

$$\psi(A) = VA \text{ and } \Psi(A) = AV$$

where V is a unitary matrix. Then

$$|\det J(\psi)| = |\det J(\Psi)| = 1.$$

Proof: The prove is straightforward, we shall consider only the map ψ. By the above definitions and the cyclic properties of the trace, we have

$$\langle \psi A, \psi B \rangle = \mathbf{Tr}\left((VA)^*VB\right) = \mathbf{Tr}\left(A^*V^*VB\right) = \langle A, B \rangle. \tag{9}$$

It is a standard result in matrix analysis [4] that for any linear operator P satisfying

$$\langle PA, PB \rangle = \langle A, B \rangle$$

we must have $|\det J(P)| = 1$. Hence, P is a unitary operator. As a result we conclude $J(\psi) = 1$.

Corollary 3 *The volume measure $d_v M$ is invariant under the linear maps ψ, Ψ, and compositions of both, e.g., $\psi^* \circ \Psi(A) = U^* A U$.*

The next two results assures that $M \xrightarrow{\varphi} (\Lambda,\, W)$ give rise to a local coordinate system.

Proposition 4 *The eigenvalues of a random normal matrix is almost surely distinct in Lebesgue measure.*

Proof: The main ingredient is Sard's theorem [7]. Consider the map $M \overset{F}{\mapsto} \prod_{i<j} |z_i - z_j||\bar{z}_i - \bar{z}_j|$, where z_i are the eigenvalues of M. By Sard's theorem the critical points of F (i.e. the normal matrices X such that $dF(X)$ is not surjective) has zero measure. By our construction, the critical points are the normal matrices with two or more equal eigenvalues. As a result, it follows that a given matrix M has almost surely distinct eigenvalues. □

A consequence of the above result is the following

Proposition 5 *There exist (almost surely) a local coordinate system in a neighborhood of a given matrix M in $\mathcal{N}$.*

Proof: Given a normal matrix M, there exist a unitary matrix $U \in \mathcal{U}(n)$ (Lie group of unitary matrices) such that

$$\Lambda = U^* M U \tag{10}$$

where Λ is a diagonal matrix of the eigenvalues of M [3].

As a consequence of Proposition 4 almost surely of eigenvalues of M are distinct, implying that the eigenvectors of M are mutually orthonormal. Hence, there are $2n$ real coordinates from the n complex eigenvalues Λ and $n^2 - n$ real implicit coordinates of the orthonormal eigenvectors U.

Hence, (Λ, W) is isomorphic to $\mathbb{R}^{2d}$. Therefore, (Λ, W) give rise to a set of local coordinates on $\mathcal{N}(n)$. □

Proposition 6 *The total differentiation of a normal matrix M is*

$$dM = d\Lambda + [dW, \Lambda] \tag{11}$$

where Λ is a diagonal matrix of the eigenvalues of M, W is a skew-Hermitian matrix with zero diagonal given by $U = e^W$.

Proof: Given the matrix M, there exists a normal matrix M_0 and a unitary matrix V such that

$$M = V M_0 V^*,$$

and $M_0 = U_0 \Lambda U_0^*$ with $\|U_0 - I\|$ small (U is close the unity).

As our ensemble is unitary invariant by Corollary 3, we may consider only M_0. Recall that the exponential map

$$W \to U = e^W$$

provides an isomorphism between the neighbors of 0_n in skew-Hermitian matrix space and I in the unitary matrix space. Hence, the total differentiation of a normal matrix M is

$$\begin{aligned}
dM &= d\left(U \Lambda U^*\right) \\
&= \left(de^W\right) \Lambda \left(e^W\right)^* + e^W \left(d\Lambda\right) \left(e^W\right)^* + e^W \Lambda \left(d\left(e^W\right)^*\right) \\
&= e^W \left(dW\right) \Lambda e^{-W} + e^W \left(d\Lambda\right) e^{-W} + e^W \Lambda e^{-W} \left(-dW\right)
\end{aligned}$$

and at $U = I$ (i. e. $W = 0_n$) we have

$$dM = (dW)\,\Lambda + d\Lambda - \Lambda dW = d\Lambda + [dW, \Lambda]\,. \tag{12}$$

In terms of matrix elements we have

$$dM_{ij} = \begin{cases} d\Lambda_{ii} & \text{if } i = j \\ (z_j - z_i)\, dW_{ij} & \text{if } i \neq j \end{cases}\,. \tag{13}$$

By definition of differential, see Eq. (13), the coordinates are

$$\frac{\partial M_{ij}}{\partial \Lambda_{kk}} = \delta_{ij}\delta_{ik} \tag{14}$$

$$\frac{\partial M_{ij}}{\partial W_{kl}} = (z_l - z_k)\,\delta_{ik}\delta_{jl} \tag{15}$$

and zero for any other partial derivative. Here, δ is the Kronecker delta function.□

Now we are ready to prove Theorem 1, which consists in realizing the transformation of the $2d = n^2 + n$ real independent coordinates (here we choose the upper triangular part) of a matrix M to the $2n$ eigenvalues coordinates $\Lambda = diag\,(z_1, \cdots, z_n)$ and the $n^2 - n$ coordinates of the skew-hermitian matrix W with zero diagonal.

Proof of theorem 1.

To compute the Jacobian of the coordinate change, we shall organize the independent variables as a vector $\overrightarrow{M}$ constructed by means of the $2d$ independent elements of the upper triangular normal matrix M ordered in the following way

$$\overrightarrow{M} = (\overbrace{M_{11}, \cdots, M_{nn}}^{n}, \overbrace{M_{12}, M_{13}, \cdots, M_{n-1n}}^{l}, \overbrace{\overline{M}_{11}, \cdots, \overline{M}_{nn}}^{n}, \overbrace{\overline{M}_{12}, \cdots, \overline{M}_{n-1n}}^{l}),$$

where $l = n(n-1)/2 = d - n$. Note that of ℓ variables are the independent lines of M organized one after the other. Therefore, we have selected $2d$ degrees of freedoms that generate the space of normal matrices.

A differential vector operator $\overrightarrow{\partial}$ is constructed with respect to the elements of the upper triangular matrix $\Lambda + W$ ordered as the vector $\overrightarrow{M}$. Thus, the differential vector operator can be written as

$$\overrightarrow{\partial}^{\,t} = (\partial_{z_1}, \cdots, \partial_{z_n}, \partial_{w_{12}}, \cdots, \partial_{w_{(n-1)n}}, \partial_{\overline{z}_1}, \cdots, \partial_{\overline{z}_n}, \partial_{\overline{w}_{12}} \cdots, \partial_{\overline{w}_{(n-1)n}})$$

where

$$\partial_z = \frac{\partial}{\partial z} \text{ and } \partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}}.$$

The Jacobian matrix of the coordinate change φ can then be written as matrix product

$$Jac\,(\varphi) = \overrightarrow{\partial} \cdot \overrightarrow{M}$$

Now, realizing the explicit computations of the derivatives with the help of (12), (14) and (15) we obtain

$$\overrightarrow{\partial} \cdot \overrightarrow{M}^t = \begin{pmatrix} I & 0_{nl} & 0_n & 0_{nl} \\ 0_{ln} & D_l & 0_{ln} & 0_{ll} \\ 0_n & 0_{nl} & I & 0_{nl} \\ 0_{ln} & 0_{ll} & 0_{ln} & \overline{D}_l \end{pmatrix}$$

where

$$D_l = diag\,(z_2 - z_1, z_3 - z_1, \cdots, z_n - z_{n-1})$$

is a diagonal matrix of order l. Hence, the Jacobian of the transformation reads

$$\det \overrightarrow{\partial} \cdot \overrightarrow{M}^t = \det D_l \det \overline{D}_l = \prod_{1 \le i < j \le n} |z_j - z_i|^2$$

□

3 Conclusion

We have realized the coordinate change from the normal matrix elements to the spectral coordinates by standard means, that is, evaluating the Jacobian of the coordinate change.

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